The $15^{\textbf{th}}$ Seminar on
Differential Equations and Dynamical Systems

24-25 June 2020, University of Guilan, Rasht, Iran

PERMANENCY IN PREDATOR-PREY MODELS OF LESLIE TYPE WITH RATIO-DEPENDENT SIMPLIFIED HOLLING TYPE-IV FUNCTIONAL RESPONSE

HASSAN QOLIZADEH AMIRABAD AND OMID RABIEIMOTLAGH

Abstract. We consider a predator-prey model of Leslie type with ratio-dependent simplified Holling type-IV functional response. First, We show that permanency of the system holds automatically for some values of parameters. Second, we establish the sufficient conditions for the global stability of interior equilibrium by constructing Lyapunov function. As a result, the permanency of the system can be solved.

1. Introduction

we study the following predator-prey system of Leslie type with ratio-dependent simplified Holling type IV functional response

$$\begin{aligned}
&\dot{x} = rx(1-\tfrac{x}{K}) - \underbrace{\left(\frac{m\frac{x}{y}}{a+\frac{x^2}{y^2}}\right)}_{p(x/y)} y \;=\; rx(1-\tfrac{x}{K}) - \tfrac{m\,xy^2}{ay^2+x^2},\\
&\dot{y} = sy(1-\tfrac{y}{hx}),\\
&x(0)>0, y(0)>0.
\end{aligned} \tag{1.1}$$

Keywords: Predator-prey of Leslie type system, Global asymptotic stability, Permanency .

AMS Mathematical Subject Classification [2010]: 03C99 , 37G10.

where x and y represent respectively prey and predator densities which grow in a logistic manner with intrinsic growth rate r and s respectively. The function $p(x/y)$ is known as ratio-dependent functional response.

2. PERMANENCY

For simplicity,we write the system in nondimensional form . Let $\tau = rt$, $\bar{x} = x/K$ and $\bar{y} = my/rK$. we obtain the following system (after eliminate the bar)

$$\begin{aligned} &\dot{x} = x(1-x) - \tfrac{nxy^2}{by^2+x^2}, \\ &\dot{y} = \delta y(\beta - \tfrac{y}{x}), \\ &x(0) > 0, y(0) > 0, \end{aligned} \tag{2.1}$$

where $n = r/m$,$b = an$, $\delta = \frac{s}{r}$ and $\beta = mh/r$.
System (2.1) has two fixed points, boundary equilibrium $E_0 = (1, 0)$ and interior equilibrium $E_1 = (x^*, \beta x^*)$ with $x^* = (1 + (b-n)\beta^2)/1 + b\beta^2$. Interior equilibrium E_1 is biologically admissible (we simply say E_1 exists) iff $x^* > 0$ or equivalently $\beta^2(n-b) < 1$. This implies that if $b \geq n$ then E_1 exists and if $0 < b < n$, then E_1 exists if $\beta\sqrt{(n-b)} < 1$.
In this section we discuss the permanency of system (1.1) which is defined as follows.

Definition 2.1. The system (2.1) is said to be permanent if there exist positive constants r and s with $0 < r \leq s$ such that for any solution $(x(t), y(t))$ with $x(0) > 0$ and $y(0) > 0$;

$$\begin{aligned} &min\{\liminf_{t\to\infty} x(t), \liminf_{t\to\infty} y(t)\} \geq r, \\ &max\{\limsup_{t\to\infty} x(t), \limsup_{t\to\infty} y(t)\} \leq s. \end{aligned}$$

In order to prove the permanency for system (2.1), we first need the following lemma.

Lemma 2.2. *(See([1]).) Solutions of system (2.1) are positive and bounded, furthermore there exists $T > 0$ such that $0 < x(t) < M$, $0 < y(t) < N$, for all $t > T$, Where $M > 1, N > M\beta$.*

We have the following result.

Lemma 2.3. *If $b > n$ then system (2.1) is permanent.*

Proof. Suppose that $b > n$, From the first equation of system (2.1) $\dot{x} > x(1 - x - \frac{n}{b})$. Hence, by the comparison theorem $\liminf_{t\to\infty} x(t) \geq 1 - \frac{n}{b} \equiv x_0$. Also, it is easy to see that $\dot{y} > \delta(\beta - \frac{2}{x_0}y)$.Thus, by the comparison theorem $\liminf_{t\to\infty} y(t) \geq \frac{\beta x_0}{2} \equiv y_0$. Let $s = max\{M, N\}$ and $r = min\{x_0, y_0\}$, from Definition 2.1 system (2.1) is permanent. □

3. Local and Global Stability of E_1

we use the following transformation which obtain locally equivalent system to system (2.1) at $E_1 = (x^*, y^*)$. Let $x = x$,$u = \frac{y}{x}$ and $p(u) = nu^2/(bu^2+1)$ Then we have

$$\begin{aligned} &\dot{x} = x(1-x) - p(u)x, \\ &\dot{u} = u(x - 1 + p(u) + \delta\beta - \delta u), \\ &x(0) > 0, u(0) > 0. \end{aligned} \tag{3.1}$$

In the rest of this section, we consider system (3.1) instead of system (2.1). Obviously, interior equilibrium $E_1 = (x^*, y^*)$ of system (2.1) is turned to $E_1 = (x^*, u^*)$ for system (3.1), where $x^* = 1 - p(\beta)$, $u^* = \beta$.
The Jacobi matrix of system (3.1) at E_1 takes the form
$J(E_1) = \begin{pmatrix} -1 + p(\beta) & (-1 + p(\beta))p^{'}(\beta) \\ \beta & \beta(-\delta + p^{'}(\beta)) \end{pmatrix} = \begin{pmatrix} -x^* & \frac{-2x^*(1-x^*)^2}{\beta^3} \\ \beta & \frac{-n\delta\beta^3 + 2(1-x^*)^2}{n\beta^2} \end{pmatrix}.$
The determinant and the Trace of $J(E_1)$ are computed as:
$Det(J(E_1)) = \beta\delta(1 - p(\beta)) = \delta\beta x^* > 0,$
$Tr(J(E_1)) = p(\beta) + \beta p^{'}(\beta) - 1 - \beta\delta = \frac{2x^{*2} - (4 + n\beta^2)x^* + 2 - n\beta^3\delta}{n\beta^2}.$

Lemma 3.1. *Let the condition $\beta^2(n - b) < 1$ hold:*
(i) If $n\delta\beta^3 < 2$ and $\beta\delta > p(\beta) + \beta p^{'}(\beta) - 1$ or $n\delta\beta^3 \geq 2$ then the interior equilibrium E_1 exists and is locally asymtotical stable.
(ii) If $n\delta\beta^3 < 2$ and $\beta\delta < p(\beta) + \beta p^{'}(\beta) - 1$ then the interior equilibrium E_1 exists and is an unstable focus or node.

Now, we introduce a Lyapunov function for the system (3.1) and use it to prove its global stability.

Theorem 3.2. *Assume that $\beta^2(n - b) < 1$, $n\delta\beta^3 > 2$ and $b > \frac{\beta^2}{16 - 8n\beta^2} > 0$, then all solutions of system (3.1) with initial condition $x_0 > 0, y_0 > 0$ satisfy*
$\liminf_{t\to\infty} x(t) = x^*, \liminf_{t\to\infty} y(t) = y^*.$

Proof. Note that, the condition of (i,ii) ensure local stability of E_1. We construct the following Lyapunov function

$$V(t) = \int_{x^*}^{x} \frac{\xi - x^*}{\xi} d\xi + \int_{\beta}^{u} \frac{p(\eta) - p(\beta)}{\eta} d\eta$$

The time derivative of V along the solutions of system (3.1) is

$$\dot{V} = -(x - x^*)^2 - (p(u) - p(\beta))(u - \beta)\frac{\delta(b\beta^2 + 1)(bu^2 + 1) - n(u + \beta)}{(b\beta^2 + 1)(bu^2 + 1)},$$

by using the conditions (ii) and (iii) we obtain

$$\delta(b\beta^2 + 1)(bu^2 + 1) - n(u + \beta) > \frac{2bu^2 - n\beta u - n\beta^2 + 2}{\beta} > 0.$$

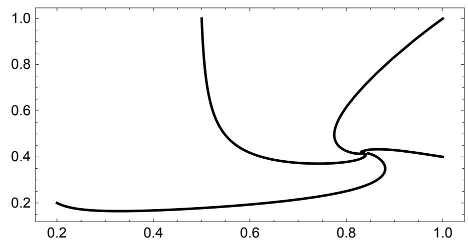

FIGURE 1. parametric plot with $(b, \beta, \delta) = (2, 0.5, 1)$ and initial conditions $(x_0, y_0) = (1, 1), (0.2, 0.2), (0.5, 1), (1, 0.4)$.All solutions converge to $E_1 = (5/6, 5/12)$ and stay away from the axis.

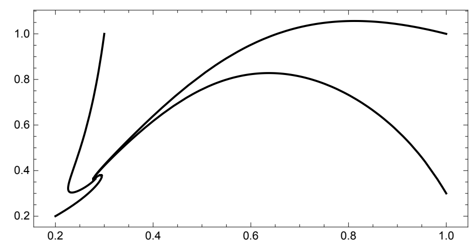

FIGURE 2. parametric plot with $(b, \beta, \delta) = (0.8, 1.3, 1)$ and initial conditions $(x_0, y_0) = (1, 1), (0.2, 0.2), (0.3, 1), (1, 0.3)$.All solutions converge to $E_1 \simeq (0.281, 0.366)$ in agreement with the theoretical results.

We know that derivation function $p^{'}(u) > 0$ for $u > 0$. Hence $\dot{V}(t) < 0$ along the solutions of system (3.1) in the first quadrant except E_1. $\square$

we present numerical simulations of system (2.1). We choose parameters $b = 2, \beta = 1/2, \delta = 1 = n$ such that $\beta^2(n - b) < 1$ and $b > n$. By Lamma 2.3 the permanency holds. The biological interpretation is that, the prey and the predator populations will survive over the long term. The simulations are shown in Fig.1. Now we choose $b = 0.8, \beta = 1.3, \delta = 1 = n$ such that $\beta^2(n - b) < 1, b < n$. The conditions of lamma 3.1 and theorem 3.2 are held. Thus, local stability implies global stability. As a result, we still have permanency. For example see Fig.2.

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DEPARTMENT OF APPLIED MATHEMATICS, UNIVERSITY OF BIRJAND, BIRJAND, IRAN
E-mail address: h.qolizadeh@birjand.ac.ir

DEPARTMENT OF APPLIED MATHEMATICS, UNIVERSITY OF BIRJAND, BIRJAND, IRAN
E-mail address: orabieimotlagh@Birjand.ac.ir